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An Expansion of the Gegenbauer Polynomial $C_n^{\mu}(xy)$

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Preface

The research presented in this report was conducted under NUSC IR/IED Project No. A70210, *Optimization of Mutually Coupled Arrays*, Principal Investigator, Dr. R. L. Streit (Code 731). The Program manager is CAPT D. F. Parrish, Chief of Naval Material (MAT 08L).

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Reviewed and Approved: 25 March 1982

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An expansion of the Gegenbauer polynomial $C_n^{\mu}(xy)$ in an orthogonal series in the polynomials $C_k^{\lambda}(x)$ with coefficients depending on y is derived. The coefficients of $C_k^{\lambda}(x)$ in the expansion are derived in a form that, by inspection, shows them to be positive for $y>1$ and $\mu \geq \lambda \geq 0$. A limiting form for these expansion coefficients is also derived. This limiting form, together

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20. (Continued)

with an apparently new formula of Mehler-Heine type, is shown to imply Sonine's second finite integral.

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An Expansion of the Gegenbauer Polynomial $C_n^\mu(xy)$

Introduction

Sonine's second finite integral [1, p. 376] may be written

$$\int_0^{\pi/2} J_\mu(x \sin\theta)\, J_\lambda(y \cos\theta) \sin^{\mu+1}\theta \cos^{\lambda+1}\theta \, d\theta$$

$$= \frac{x^\mu y^\lambda J_{\mu+\lambda+1}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^{\mu+\lambda+1}} \qquad (1)$$

for all complex x and y, and is valid when both $\mathrm{Re}(\mu)>-1$ and $\mathrm{Re}(\lambda)>-1$. At least two proofs of this result are known. One involves expanding the integral in powers of x and y; the other involves integration over subsets of the surface of the unit sphere in $\mathbf{R}^3$. Both are given in [1].

For the case of real μ and λ, a third proof is given here that depends in an essential way on the identity (7). In this connection, the particular form of the coefficients $a_{k,n}(y)$ is important; that is, the easily derived identity (10) does not seem to be at all useful, but the identity (8) is exactly what is needed. It facilitates the investigation of the limiting form (27) of $a_{k,n}(y)$ as n tends to infinity. The identity (8) is apparently new; however, the special case of $y = 1$ was known to Gegenbauer.

Equation (8) is interesting in another regard as well. A simple inspection suffices to prove that $a_{k,n}(y) > 0$ for all n and k whenever $y>1$ and $\mu \geq \lambda > 0$. The coefficients remain positive in the two limiting cases $\mu > 0$, $\lambda = 0$ and $\mu = \lambda = 0$, as can be seen from (18) through (21). In fact, it was only this positivity result that the author originally sought.

The result (3) of Mehler-Heine type is apparently new. It is needed to prove (1) by our methods. It has additional interest in that it duplicates Szegö's result (2) simply by setting $y = 0$. Since Szegö's proof of (2) may very nearly be lifted verbatim to prove (3), it is perhaps surprising that he does not mention (3) in [2]. The special case (4) involving Chebyshev polynomials is particularly striking and seems to be new also.

Derivations and Results

Let α and β be arbitrary real numbers. For any complex number x, the Mehler-Heine theorem states that

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{x}{n}\right) = (x/2)^{-\alpha} J_\alpha(x), \tag{2}$$

where $J_\alpha(x)$ is the Bessel function of the first kind of order α [1, §3.1(8); 2 (1.71.1)]. A straightforward proof of (2) can be found in Szegö [2, Theorem 8.1.1]. Szegö's proof can be readily modified to show that

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right) = (\tfrac{1}{2}\sqrt{x^2-y^2}\,)^{-\alpha}\, J_\alpha(\sqrt{x^2-y^2}\,) \tag{3}$$

for all complex x and y. Like the Mehler-Heine result, this formula holds uniformly for x and y in every bounded region of the complex plane. The special case $\alpha=\beta=-1/2$ gives the interesting result

$$\lim_{n\to\infty} T_n\left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right) = \cos\sqrt{x^2-y^2}\ , \tag{4}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind [2, (4.1.7)]. This follows from (3) by using Stirling's formula and the well known result [2, (1.71.2)]

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z. \tag{5}$$

We will need another special case of the general result; specifically, for $\mu>-1$,

$$\lim_{n\to\infty} \frac{n^{1-2\mu}}{2\mu}\, C_n^\mu\left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right) = \sqrt{\pi/2}\ \frac{J_{\mu-\frac{1}{2}}(\sqrt{x^2-y^2}\,)}{2^\mu\,\Gamma(\mu+1)\,(\sqrt{x^2-y^2}\,)^{\mu-\frac{1}{2}}}\ , \tag{6}$$

where $C_n^\mu(x)$ are the ultraspherical, or Gegenbauer, polynomials [1, (4.7.1)]. (Szegö uses the notation $P_n^{(\mu)}(x)$ instead of $C_n^\mu(x)$.)

We derive Sonine's second finite integral by finding an alternate form for the left-hand side of (6). This requires the following result. For $\mu\geqslant\lambda>0$, the coefficients $a_{k,n}(y)$ in the expansion

$$C_n^\mu(xy) = \sum_{k=0}^{[n/2]} a_{k,n}(y)\, C_{n-2k}^\lambda(x),\ n=0, 1, 2, \ldots \tag{7}$$

are given explicitly by

$$a_{k,n}(y) = (n-2k+\lambda)\,(\mu)_{n-k} \sum_{m=0}^{k} \frac{(\mu-\lambda+m)_{k-m}(y^2-1)^m\, y^{n-2m}}{m!\,(k-m)!\,(\lambda)_{n-k-m+1}}, \tag{8}$$

where we take $0^0 = 1$ and $(0)_0 = 1$ whenever they occur. Setting $y = 1$ in (8) gives

$$a_{k,n}(1) = \frac{(n-2k+\lambda)\,(\mu)_{n-k}\,(\mu-\lambda)_k}{k!\,(\lambda)_{n-k+1}}, \tag{9}$$

which is due to Gegenbauer [2, (4.10.27)]. Furthermore, for real $y > 1$ and $\mu \geqslant \lambda > 0$, the coefficients $a_{k,n}(y)$ are all positive as can be seen by inspection in (8).

The formula (8) is derived as follows. Let $\mu \geqslant \lambda > 0$. In the expression [2, (4.7.31)]

$$C_n^{\mu}(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(\mu)_{n-m}}{m!\,(n-2m)!} (2x)^{n-2m},$$

we replace x with xy, substitute

$$\frac{(2x)^{n-2m}}{(n-2m)!} = \sum_{s=0}^{\left[\frac{n-2m}{2}\right]} \frac{(n-2m+\lambda-2s)}{s!\,(\lambda)_{n-2m-s+1}} C^{\lambda}_{n-2m-2s}(x),$$

and collect terms to get

$$a_{k,n}(y) = \sum_{m=0}^{k} (-1)^m \frac{(n-2k+\lambda)\,(\mu)_{n-m}\, y^{n-2m}}{m!\,(k-m)!\,(\lambda)_{n-m-k+1}}, \tag{10}$$

$$= y^{n-2k}\,(n-2k+\lambda)\,Q_k(2y^2-1), \tag{11}$$

where Q_k is a polynomial defined for general complex argument u by

$$Q_k(u) = \sum_{m=0}^{k} \frac{(-1)^m\,(\mu)_{n-m}}{m!\,(k-m)!\,(\lambda)_{n-m-k+1}} \left(\frac{u+1}{2}\right)^{k-m}. \tag{12}$$

For arbitrary α and β, the Jacobi polynomial of degree $k \geqslant 0$ can be written

$$P_k^{(\alpha,\beta)}(u) = \sum_{m=0}^{k} (-1)^m \frac{(k+\alpha+\beta+1)_{k-m}\,(k-m+\beta+1)_m}{m!\,(k-m)!} \left(\frac{u+1}{2}\right)^{k-m}, \tag{13}$$

which follows from [2, (4.21.2)] using the identity [2,(4.1.3)]. Setting $\alpha = \mu - \lambda - 1$ and $\beta = \lambda + n - 2k$ in (13) shows that

$$Q_k(u) = \frac{(\mu)_{n-k}}{(\lambda)_{n-k+1}} \quad P_k^{(\mu-\lambda-1,\, \lambda+n-2k)}(u). \tag{14}$$

Expanding the Jacobi polynomial in (14) using [2, (4.3.2)]

$$P_k^{(\alpha,\beta)}(u) = \sum_{m=0}^{k} \frac{(1+\alpha)_k\,(1+\beta)_k}{m!\,(k-m)!\,(1+\alpha)_m\,(1+\beta)_{k-m}} \left(\frac{u-1}{2}\right)^m \left(\frac{u+1}{2}\right)^{k-m}, \tag{15}$$

and substituting $u = 2y^2-1$ gives

$$Q_k(2y^2-1) = (\mu)_{n-k} \sum_{m=0}^{k} \frac{(\mu-\lambda+m)_{k-m}(y^2-1)^m\, y^{2k-2m}}{m!\,(k-m)!\,(\lambda)_{n-k-m+1}}. \tag{16}$$

Thus (16) and (11) establish (8).

Two limiting cases of (7) are easily derived from [2, (4.7.8)]

$$\lim_{\lambda \to 0} \frac{n}{2\lambda}\, C_n^{\lambda}(x) = T_n(x), \quad n \geqslant 1, \tag{17}$$

and are worth recording. Thus, for $\mu > 0$,

$$C_n^{\mu}(xy) = {\sum_{k=0}^{[n/2]}}' b_{k,n}(y)\, T_{n-2k}(x), \quad n = 0, 1, 2, \ldots, \tag{18}$$

where

$$b_{k,n}(y) = 2(\mu)_{n-k} \sum_{m=0}^{k} \frac{(\mu+m)_{k-m}(y^2-1)^m\, y^{n-2m}}{m!\,(k-m)!\,(n-k-m)!}, \tag{19}$$

and

$$T_n(xy) = {\sum_{k=0}^{[n/2]}}' c_{k,n}(y)\, T_{n-2k}(x), \quad n = 1, 2, 3, \ldots, \tag{20}$$

where

$$c_{k,n}(y) = n(n-k-1)! \sum_{m=0}^{k} \frac{(m)_{k-m}(y^2-1)^m\, y^{n-2m}}{m!\,(k-m)!\,(n-k-m)!}. \tag{21}$$

The notation Σ' means that 1/2 the last term in the sum is taken if n is even, and all of it is taken if n is odd. Note that inspection shows that $y>1$ implies $b_{k,n}(y)$ and $c_{k,n}(y)$ are positive.

Sonine's second finite integral is now derived from (6). Fix x and y. Let $N=[n/2]$. From (7)

$$\frac{n^{1-2\mu}}{2\mu} C_n^{\mu}\left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right)$$

$$=\frac{1}{1+N}\sum_{k=0}^{N}\left\{\frac{\lambda n^{1-2\mu}(1+N)}{\mu(n-2k)^{1-2\lambda}} a_{k,n}\left(\frac{1}{\cos\frac{y}{n}}\right)\right\}\left\{\frac{(n-2k)^{1-2\lambda}}{2\lambda} C_{n-2k}^{\lambda}(\cos\frac{x}{n})\right\}$$

$$=\int_0^1 f_n(1-\zeta)\, g_n(1-\zeta)\, d\zeta, \qquad (22)$$

where we have defined for $0\leqslant\zeta\leqslant 1$,

$$f_n(1-\zeta)=\sum_{k=0}^{N}\frac{\lambda n^{1-2\mu}(1+N)}{\mu(n-2k)^{1-2\lambda}} a_{k,n}\left(\frac{1}{\cos\frac{y}{n}}\right) \chi_{E_k}(1-\zeta),$$

$$g_n(1-\zeta)=\sum_{k=0}^{N}\frac{(n-2k)^{1-2\lambda}}{2\lambda} C_{n-2k}^{\lambda}(\cos\frac{x}{n})\, \chi_{E_k}(1-\zeta),$$

and χ_{E_k} is the characteristic (indicator) function of the interval

$$E_k=\begin{cases}\left[\dfrac{k}{N+1}, \dfrac{k+1}{N+1}\right), & k=0, 1, \ldots, N-1,\\[2ex] \left[\dfrac{k}{N+1}, \dfrac{k+1}{N+1}\right], & k=N.\end{cases}$$

It can be verified that $\chi_{E_k}(2k/n)=1$ for $k=0, 1, \ldots, N$.

Assume for the moment that both $|f_n(\zeta)|$ and $|g_n(\zeta)|$ are bounded above by integrable functions of ζ. To do this, it will be seen that we must restrict attention to $\lambda>-1/2$, $\mu>-1/2$, $\mu>\lambda$, so that the integral [2, (1.7.4)]

$$\int_0^1 \zeta^{2\lambda}(1-\zeta^2)^{\mu-\lambda-1}d\zeta=\frac{\Gamma(\lambda+½)\,\Gamma(\mu-\lambda)}{2\Gamma(\mu+½)} \qquad (23)$$

will be finite. If $f = \lim f_n$ and $g = \lim g_n$, the bounded convergence theorem [3, p. 110] implies

$$\lim_{n\to\infty} \frac{n^{1-2\mu}}{2\mu} C_n^{\mu}\left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right) = \int_0^1 f(1-\zeta)\, g(1-\zeta)\, d\zeta. \tag{24}$$

Let ζ in (0,1) be rational. Then $1-\zeta = 2k/n$ for sufficiently large k and n, so that

$$\begin{aligned} g(1-\zeta) &= \lim_{n\to\infty} g_n(1-\zeta) \\ &= \lim_{\substack{n\to\infty \\ 1-\zeta=2k/n}} \frac{(n-2k)^{1-2\lambda}}{2\lambda} C_{n-2k}^{\lambda}(\cos\frac{x}{n}) \\ &= \sqrt{\pi/2}\ \frac{J_{\lambda-\frac{1}{2}}(\zeta x)}{2^{\lambda}\Gamma(\lambda+1)\,(\zeta x)^{\lambda-\frac{1}{2}}} \end{aligned} \tag{25}$$

with the last step following immediately from (6). Thus, (25) holds for all ζ in [0,1] by continuity. Similarly, from (8) and for all ζ rational in (0,1),

$$\begin{aligned} f(1-\zeta) &= \lim_{n\to\infty} f_n(1-\zeta) \\ &= \lim_{\substack{n\to\infty \\ 1-\zeta=2k/n}} \frac{\lambda n^{1-2\mu}(1+N)}{\mu(n-2k)^{1-2\lambda}}\ a_{k,n}\left(\frac{1}{\cos\frac{y}{n}}\right) \\ &= \lim_{\substack{n\to\infty \\ 1-\zeta=2k/n}} \sum_{m=0}^{k} \frac{(\zeta+\frac{\lambda}{n})\,(\mu+1)_{n-k-1}\, n^{-2(\mu-\lambda-1)}\left(\frac{1}{n}+\frac{N}{n}\right)(\mu-\lambda+m)_{k-m}\sin^{2m}\frac{y}{n}}{\zeta^{1-2\lambda}\cos^{n}\frac{y}{n}\ m!\,(k-m)!\,(\lambda+1)_{n-k-m}}. \end{aligned} \tag{26}$$

Interchange the limit and the summation, and evaluate the limit of the mth term (convert Pochhammer symbols to Gamma functions, apply Stirling's formula, and use $k(n-k) = (1-\zeta^2)\, n^2/4$) to obtain

$$f(1-\zeta) = \sum_{m=0}^{\infty} \frac{\zeta^{2\lambda}(1-\zeta^2)^{\mu-\lambda-1}}{2^{2\mu-2\lambda-1}}\ \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}\ \frac{(\frac{1}{2}y\sqrt{1-\zeta^2})^{2m}}{m!\,\Gamma(\mu-\lambda+m)}$$

$$= \frac{\zeta^{2\lambda}(1-\zeta^2)^{\mu-\lambda-1}}{2^{2\mu-2\lambda-1}}\ \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}\ \frac{I_{\mu-\lambda-1}(y\sqrt{1-\zeta^2})}{(\frac{1}{2}y\sqrt{1-\zeta^2})^{\mu-\lambda-1}}, \tag{27}$$

where $I_\nu(z)$ denotes the modified Bessel function of the first kind of order ν (see [1, §3.7(2)]. We must require $\mu > \lambda$ in (27) to have convergence. Continuity again assures that (27) holds for all ζ in (0,1). Now, interchanging the limit and the sum

was valid because an upper bound for the total sum can be found. Since the absolute value of the m^{th} term in (26) is bounded by

$$B \quad \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \quad \frac{(\tfrac{1}{2}|y|\sqrt{1-\zeta^2})^{2m}}{m!\,\Gamma(\mu-\lambda+m)} ,$$

where

$$B = \frac{\zeta+\frac{\lambda}{n}}{\zeta^{1-2\lambda}} \quad \frac{n^{-2(\mu-\lambda-1)}}{n^{2m}|\cos^n \frac{y}{n}|} \quad \left(\frac{1-\zeta^2}{4}\right)^{-m} \frac{\Gamma(k+\mu-\lambda)}{\Gamma(k-m+1)} \quad \frac{\Gamma(n-k+\mu)}{\Gamma(n-k+\lambda+1-m)}$$

$$\cong \zeta^{2\lambda} \left(\frac{1-\zeta^2}{4}\right)^{\mu-\lambda-1}, \quad n \to \infty,$$

the total sum in (26) is bounded by

$$F(\zeta) = L\zeta^{2\lambda}(1-\zeta^2)^{\mu-\lambda-1} \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \sum_{m=0}^{\infty} \frac{(\tfrac{1}{2}|y|\sqrt{1-\zeta^2})^{2m}}{m!\,\Gamma(\mu-\lambda+m)} < \infty \tag{28}$$

for some constant L independent of ζ. The *series* in (28) is a continuous function of ζ on [0,1] if $\mu > \lambda$. Hence, from (23), $F(\zeta)$ is an integrable function that bounds $|f_n(\zeta)|$ for all n.

From (24), (25), and (27) we have

$$\lim_{n\to\infty} \frac{n^{1-2\mu}}{2\mu} C_n^{\mu} \left(\frac{\cos\frac{x}{n}}{\cos\frac{y}{n}}\right)$$

$$= \frac{\sqrt{\pi/2}}{2^{\mu}\Gamma(\mu+1)\,x^{\lambda-\frac{1}{2}}\,y^{\mu-\lambda-1}} \int_0^1 \zeta^{\lambda+\frac{1}{2}}(\sqrt{1-\zeta^2})^{\mu-\lambda-1} J_{\lambda-\frac{1}{2}}(\zeta x)\, I_{\mu-\lambda-1}(y\sqrt{1-\zeta^2})\, d\zeta$$

$$= \frac{\sqrt{\pi/2}}{2^{\mu}\Gamma(\mu+1)} \quad \frac{J_{\mu-\frac{1}{2}}(\sqrt{x^2-y^2})}{(\sqrt{x^2-y^2})^{\mu-\frac{1}{2}}} ,$$

with the last equation from (6). Substituting $\zeta = \sin\theta$ and $y = iy'$ in the last two formulas, and setting

$$\mu' = \lambda - \tfrac{1}{2} > -1 \quad \text{and} \quad \lambda' = \mu-\lambda-1 > -1 \tag{29}$$

yields Sonine's second finite integral (1). The only thing left to prove is that $|g_n(\zeta)|$ is bounded by an integrable function on [0,1]. Szegö's argument [2, p. 192] in the

proof of (2) can be modified easily to show $|g_n(\zeta)|$ is bounded by a constant.

Conclusions

The special case $\mu = \lambda$ in (27) may merit further study. In the more restrictive case $\mu = \lambda = 0$, it is known that [4, (871.2)]

$$\cos\sqrt{x^2-y^2} - \cos x = y\int_0^{\pi/2} I_1(y\cos\theta)\cos(x\sin\theta)\,d\theta.$$

However, we do not pursue this here.

Another question that we do not investigate here is the expansion

$$P_n^{(\alpha,\beta)}(xy) = \sum_{k=0}^{n} A_{k,n}(y)\,P_k^{(\gamma,\delta)}(x).$$

It would seem difficult to obtain a form for $A_{k,n}(y)$ from which it is directly evident which conditions imply $A_{k,n}(y) > 0$.

The proof of (1) presented here was intentionally restricted to real μ and λ. However, it is not hard to see from (23) and (29) that the proof can be carried out for complex μ and λ provided appropriate remarks are made in appropriate places about the complex case. If such remarks are made, our derivation proves (1) for $\mathrm{Re}(\mu) > -1$ and $\mathrm{Re}(\lambda) > -1$. Divergence of (23) is seen to be the *cause* of the restrictions on μ and λ.

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